

# Duality between normal and superconducting junctions of multiple quantum wires

Sourin Das<sup>1,\*</sup> and Sumathi Rao<sup>2,†</sup><sup>1</sup>Centre for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India<sup>2</sup>Harish-Chandra Research Institute, Allahabad 211 019, India

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We study junctions of single-channel spinless Luttinger liquids using bosonization. We generalize earlier studies by allowing the junction to be superconducting and find new charge nonconserving low-energy fixed points. We establish the existence of  $g \leftrightarrow 1/g$  duality (where  $g$  is the Luttinger liquid parameter) between the charge-conserving (normal) junction and the charge nonconserving (superconducting) junction by evaluating and comparing the scaling dimensions of various operators around the fixed points in both the normal and superconducting sectors of the theory. For the most general two-wire junction, we show that there are two conformally invariant one-parameter families of fixed points which are also connected by a duality transformation. We also show that the stable fixed point for the two-wire superconducting junction corresponds to the situation where the crossed Andreev reflection (an incoming electron is transmitted as an outgoing hole) is perfect between the wires. For the three-wire junction, we study, in particular, the superconducting analogs of the chiral  $\mathbf{D}_p$  and the disconnected fixed points obtained earlier in the literature in the context of charge-conserving three-wire junctions. We show that these fixed points can be stabilized for  $g < 1$  (repulsive electrons) within the superconducting sector of the theory which makes them experimentally relevant.

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## I. INTRODUCTION

Recently, Y junctions of several quasi-one-dimensional (1D) quantum wires (QW) have been realized experimentally in single-walled carbon nanotubes.<sup>1,2</sup> Junctions of this kind are of importance for potential application in the fabrication of quantum circuitry. Theoretically, junctions of QW have been studied from several points of view<sup>3–26</sup> using bosonization, weak interaction renormalization-group (WIRG) methods, conformal field theory, and functional renormalization-group methods. The junction has also been variously taken to be enclosing a flux, having a resonant level, having a Kondo spin, and having a superconductor using one or the other techniques mentioned above.

A comprehensive study of the junctions of three QW enclosing magnetic flux was carried out by Chamon and co-workers,<sup>6,7</sup> where the wires were modeled as single-channel spinless Luttinger liquids (LL), and conformally invariant charge-conserving boundary conditions were identified in terms of boundary bosonic fields which had correspondence with a host of fixed points in the theory. However, superconducting junctions of multiple 1D QW have not been studied in the past for the case of arbitrarily strong electron-electron interactions.

In this paper, we study transport across multiple wires connected to a superconductor as depicted in Fig. 1. In the subgap region, normal reflection and transmission of the electrons cannot occur since charges can enter and exit the superconductor only as a Cooper pair. But due to the proximity effect, two new processes can occur. One is the phenomenon of Andreev reflection (AR) in which an electron-like quasiparticle incident on *normal-superconductor* (NS) junction is reflected back as a hole along with the transfer of two electrons into the superconductor as a Cooper pair. The second even more interesting process is “crossed Andreev reflection (CAR),”<sup>27–30</sup> whereby an electron from one wire

pairs with an electron from another wire to form a Cooper pair and jumps into the superconductor, emitting a hole in the second wire. (Note that for a singlet superconductor, the two electrons have opposite spins.) This can take place provided that the distance between the two wires  $L$  is less than or equal to the phase coherence length of the superconductor. Thus, for an incident electron, holes are either reflected or transmitted across the junction, and total current conservation is taken care of by the Cooper pairs jumping into the superconductor. However, as far as the multiple wire system is concerned, current is not conserved. The system is modeled as several 1D LL connected to a superconducting junction. We assume that the width of the superconductor between any two wires  $L \geq a$ , where  $a$  is the phase coherence length of the superconductor. For simplicity, we assume that the superconductor is a singlet. Thus spin is conserved in transport across the superconductor, and we can confine our

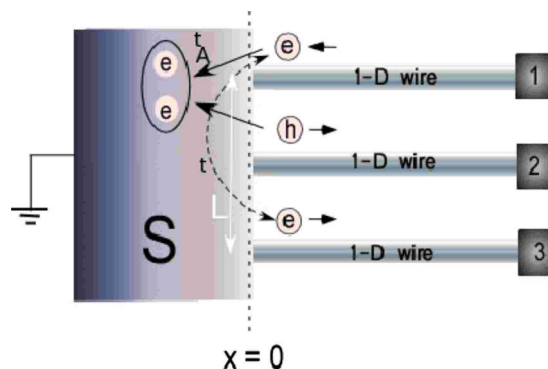


FIG. 1. (Color online) Multiple wires connected to a superconducting junction within the phase coherence length  $a$  ( $L < a$ ) of the superconductor. The processes corresponding to an incident electron on one wire undergoing CAR (amplitude represented as  $t_A$ ) and direct transmission (amplitude represented as  $t$ ) to a different wire across the superconductor are depicted in the figure.

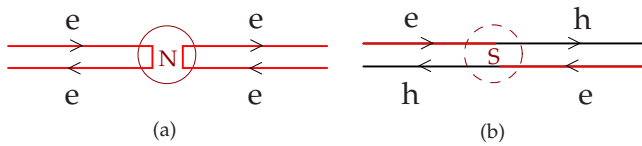


FIG. 2. (Color online) Stable fixed points of (a) normal junction (electron is completely reflected) and (b) superconducting junction (electron is perfectly transmitted as a hole).

study to spinless LL. For this system, we see that the superconductor can be modeled simply as a (charge-violating) boundary condition on the bosons in the wire. We also find a rich fixed-point structure that generalizes the earlier structure of fixed points found when multiple wires are connected to a normal junction.

The superconductor explicitly violates charge conservation at the boundary, thereby it allows for a generalization of the study of Chamon *et al.*<sup>6</sup> to the charge nonconserving sector. We find that there exists a “normal junction–superconducting junction (N-S)” duality given by  $g \leftrightarrow 1/g$  ( $g$  is the LL parameter) between the charge-conserving (normal) and the charge nonconserving (superconducting) sectors of the theory for junctions of any number of QW. As a consequence of this duality, many of the fixed points that were unstable for the normal junction for  $g < 1$  turn out to have stable superconducting analogs. The stability of the fixed points mentioned here is calculated with respect to perturbations which are within the normal sector if the fixed point is in the normal sector and within the superconducting sector for the fixed point in the superconducting sector. The main results obtained in this paper in the context of two-wire and three-wire junctions are:

(a) For the most general two-wire junction, we show that there are two conformally invariant one-parameter families of fixed points which are connected to one another via a duality transformation. We also show that the normal sector and the superconducting sector of the theory correspond to two distinct points on each of two one-parameter families of fixed points. Hence other than these special points on the two one-parameter families, loosely speaking, the rest of the fixed points represent seminormal (semisuperconducting) junction. We find that the stable fixed point within the superconducting sector of the theory corresponds to a situation where an incoming electron is completely transmitted as an outgoing hole, as shown in Fig. 2(b). This is the CAR. This fixed point is shown to be dual to the unstable connected (perfectly transmitting) fixed point of a two-wire normal junction due to the N-S duality.

(b) For the three-wire junction, we restrict our study to the special cases of normal and superconducting sectors. Within each sector, the theory of the three-wire junction effectively reduces to the most general theory of the two-wire junction as in both cases; it is a theory of two independent bosonic fields. For the three-wire junction, out of the three independent bosonic fields, one is pinned either by the charge-conserving (normal) boundary condition or by the charge nonconserving (superconducting) boundary condition leaving behind only two independent fields. Hence for the three-wire superconducting junction also, one gets two confor-

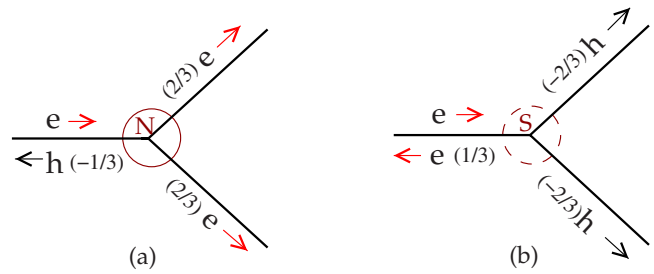


FIG. 3. (Color online) (a)  $\mathbf{D}_P$  fixed point for the normal junction.  $2/3$  charge is transmitted on each of the other wires and  $-1/3$  charge is reflected and (b)  $\mathbf{SD}_P$  fixed point for the superconducting junction.  $-2/3$  charge is transmitted on each of the other wires and  $1/3$  charge is reflected. Note that we have considered incoming electrons only along one of the wires.

mally invariant one-parameter families of fixed points, which are connected to one another via a duality transformation. Of all these fixed points for the system of a superconducting three-wire junction, we shall mainly focus on two which are of interest both theoretically and experimentally:

(i)  $\mathbf{SD}_P$  fixed point. This fixed point represents a junction with  $Z_3$  symmetry between the three wires having maximal CAR between any of the two wires. In other words, this is a fixed point where an incoming electron has nonzero components on all three wires as outgoing states.  $-2/3$  of the charge is transmitted on the two other wires (hole transmission) and  $1/3$  of the charge is backscattered (electron reflection). Note that the net change in charge at the boundary is  $e - (-2/3 - 2/3)e + (1/3)e = 2e$ . This can be identified as the charge nonconserving analog of the  $\mathbf{D}_P$  fixed point found in Ref. 7. The  $\mathbf{SD}_P$  fixed point is shown to be stable for  $g < 1/3$  within the superconducting sector and is identified as dual of the charge-conserving  $\mathbf{D}_P$  fixed point via the N-S duality. These fixed points are shown in Fig. 3.

(ii)  $\mathbf{S}\chi_{\pm}$  fixed point. These two fixed points,  $\mathbf{S}\chi_{+}$  and  $\mathbf{S}\chi_{-}$ , represent a superconducting three-wire junction with maximally asymmetric interwire CAR with broken time-reversal symmetry. An incoming electron along wire 1 is transmitted as a hole in wire 2, and so on, cyclically, as shown in Fig. 3(b) (or the other way around). They are the superconducting analogs of the chiral fixed points [Fig. 3(a)] found earlier.<sup>5,7,9</sup> Unlike their charge-conserving analogs, these fixed points are stable for  $1/3 < g < 1$ . As can be seen from the stability window of  $\mathbf{S}\chi_{\pm}$ , these are the most relevant fixed points from the experimental point of view as they can be stabilized even for a very weakly interacting ( $g \lesssim 1$ ) electron gas provided the charge-conserving perturbations are weak enough.

An extensive study of the renormalization-group evolution of several wires connected to a superconductor was carried out very recently by Das *et al.*,<sup>10,11</sup> where conductances were studied in the Landauer-Buttiker language of transmission and reflection of electrons. Interactions were taken into account perturbatively using the WIRG method. But for arbitrarily strong interelectron interactions, one needs to use bosonization. Also, since the WIRG procedure is essentially a one-particle approach, it could only access those fixed points that could be expressed linearly in terms of fermions.

To access other fixed points, one needs to use the technique of bosonization. Note that some of the fixed points obtained from the fermionic WIRG method can be identified with some of the fixed points obtained using bosonization by taking the  $g$  close to unity limit, but in general this is not true. In this paper, our aim is a comprehensive study of the system of multiple wires connected to a superconductor and to identify the various fixed points of the system, only some of which were obtained in the earlier approach.

In what follows, we first describe bosonization of superconducting junction of  $N$  number of LL wires in Sec. II. In Secs. II A and II C, we apply this method to single-wire, two-wire, and three-wire junctions and calculate scaling dimensions of various operators in the theory. In Sec. II D, we give an expression for conductance and calculate it for various fixed points obtained in Secs. II B and II C. Finally, we conclude with a discussion on general issues related to the physics of LL junctions in Sec. III.

## II. BOSONIZATION OF THE SUPERCONDUCTING JUNCTION OF LL QW

The (spinless) electron field can be written in terms of bosonic fields as

$$\psi(x) = F_O e^{ik_F x} e^{i[\theta(x)+\phi(x)]} + F_I e^{-ik_F x} e^{i[\phi(x)-\theta(x)]},$$

where  $F_O$  and  $F_I$  are Klein factors for the outgoing and incoming fields, respectively,  $\phi(x)$  and  $\theta(x)$  are the dual bosonic fields, and  $k_F$  is the Fermi momentum. The wires are modeled as spinless LL on a half line ( $x > 0$ ); i.e., here we use a folded basis for describing the junction such that all the wires lie between  $x=0$  and  $\infty$  and the junction is positioned at  $x=0$ . Hence the action is given by

$$\mathbf{S} = \int d\tau \int_0^\infty dx \sum_{i=1}^N \left[ \frac{1}{\pi} \dot{\phi}_i \theta'_i + \frac{v}{2\pi} \left\{ g(\phi'_i)^2 + \frac{1}{g}(\theta'_i)^2 \right\} \right], \quad (1)$$

where prime (dot) stands for spatial (time) derivative,  $\phi_i(x, t) = (\phi_{iO} + \phi_{iI})/2$  and  $\theta_i(x, t) = (\phi_{iO} - \phi_{iI})/2$ ;  $\dot{\phi}_i = (v/g)\theta'_i$  and  $\dot{\theta}_i = (vg)\phi'_i$ .  $\phi_{iO}$  and  $\phi_{iI}$  are the chiral outgoing and incoming bosonic fields.

The action can also be written in terms of purely the  $\phi_i$  fields or the  $\theta_i$  fields and as is well known, the two actions are identical with the replacement of  $g \leftrightarrow 1/g$ . The total densities and the currents in each wire can also be written in terms of the incoming and outgoing fields: the density  $\rho = \rho_O + \rho_I$ , with  $\rho_{O/I} = \pm(1/2\pi)\phi'_{O/I}$ , and the current  $j = j_O - j_I$ , with  $j_{O/I} = \pm v_F(1/2\pi)\phi'_{O/I}$ . To complete the theory, the action needs to be augmented by a boundary condition at the origin which represents the physics at the junction.

Now following the method we used in Ref. 9, it is possible to represent the junction in terms of a splitting matrix; i.e., we connect the incoming and the outgoing current fields as  $j_{iO}|_{x=0} = C_{ik} j_{kI}|_{x=0}$  (Ref. 31) through a current splitting matrix,  $C_{ik}$ . For charge-conserving fixed points, the net current flowing into the junction must be zero. Hence all charge-conserving fixed points must satisfy the constraint that  $\sum_i j_i$

= 0. In terms of the bosonic fields, this implies that  $\sum_i \phi'_{iO} = -\sum_i \phi'_{iI}$ . Now we can write a field splitting boundary condition as

$$\phi_{iO} = C_{ij} \phi_{jI}, \quad (2)$$

which is consistent with  $j_{iO} = C_{ik} j_{kI}$  and  $\sum_i \phi'_{iI} = -\sum_i \phi'_{iO}$ . While writing the field splitting relations in the above form, we have ignored possible integration constants, inclusion of which makes no difference to the evaluation of scaling dimensions of operators around various fixed points. Hence, the current splitting and the field splitting matrices are taken to be the same.

For the boundary condition or equivalently for the matrix  $C_{ij}$  to represent a fixed point, it should not flow under renormalization group (RG). This means that it has to be scale invariant or equivalently in 1+1 dimensions, conformally invariant. Here, this condition simply means that the trace of the energy-momentum tensor of the bosonic fields has to vanish at the boundary ( $x=0$ ). This yields the following condition:<sup>32,33</sup>

$$\sum_{i=1}^m \dot{\phi}_i \varphi'_i |_{x=0} = 0, \quad (3)$$

where  $\varphi_i$  are the mutually independent fields such that for a junction of  $N$ -wire system with  $N'$  constraints,  $m = N - N'$ . Hence, for two or more fields coupled to the junction, one can have mixed boundary conditions, besides the Dirichlet ( $\phi_i=0$ ) and Neumann ( $\phi'_i=0$ ) boundary conditions. For  $m=2$  (which is the case for the most general two-wire junction or the three-wire case in either the purely charge conserving or the purely superconducting limit), there are two independent families of solutions possible to the above equation given by

$$\varphi'_1 = -a\varphi_2, \quad \varphi'_2 = a\varphi_1, \quad (4)$$

$$\dot{\varphi}_1 = -a\dot{\varphi}_2, \quad \varphi'_2 = a\varphi'_1, \quad (5)$$

where  $a$  is a real constant, independent of  $x$  and  $t$ . In terms of the  $2 \times 2$  field splitting matrices, it is easy to check that this is equivalent to taking

$$C_{\theta_1} = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}, \quad C_{\theta_2} = \begin{pmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{pmatrix}, \quad (6)$$

where  $c_i = \cos \theta_i$  and  $s_i = \sin \theta_i$  are real parameters and  $C_{\theta_1}$  and  $C_{\theta_2}$  are the field splitting matrices at the junction for the  $\varphi$  fields. The two family of solutions are connected via duality transformation, which we call  $\mathbf{D}_{\theta_1\theta_2}$ .  $\mathbf{D}_{\theta_1\theta_2}$  duality can be accomplished by either  $\varphi_1 \leftrightarrow \vartheta_1$  or  $\varphi_2 \leftrightarrow \vartheta_2$ . For the two-wire system,  $\varphi_i$  and  $\vartheta_i$  can be identified with the  $\phi_i$  and  $\theta_i$ , respectively, with  $i$  being the wire index, and  $C_{\theta_1}$  and  $C_{\theta_2}$  can be identified with the current splitting matrices for the two-wire system. For the three-wire case,  $\varphi_i$  and  $\vartheta_i$  have to be taken to be linear combinations of  $\phi_i$ s and  $\theta_i$ s after imposing normal and superconducting boundary conditions. For these one-parameter families it turns out that the incoming and outgoing (bosonic) boundary fields satisfy the bosonic commutation relations of the bulk given by  $[\phi_{O/I}(x), \phi_{O/I}(x')]$

$= \pm i\pi \operatorname{sgn}(x-x')$ ; so imposing bosonic commutation relations gives no new constraints.

The boundary conditions may also be written in terms of the Bogoliubov transformed free bosonic fields,  $\tilde{\phi}_{iO/I}$  which are defined as<sup>9</sup>

$$\phi_{iO} = \frac{1}{2\sqrt{g}}[(g+1)\tilde{\phi}_{iO} - (g-1)\tilde{\phi}_{iI}], \quad (7)$$

$$\phi_{iI} = \frac{1}{2\sqrt{g}}[-(g-1)\tilde{\phi}_{iO} + (g+1)\tilde{\phi}_{iI}]. \quad (8)$$

For the tilde fields, the boundary condition  $\phi_{iO} = C_{ij}\phi_{jI}$  for the  $N$ -wire junction becomes

$$\tilde{\phi}_{iO} = R_{ij}\tilde{\phi}_{jI},$$

with

$$R = \frac{[(g+1)C + (g-1)\mathbb{I}]}{[(g-1)C + (g+1)\mathbb{I}]} \quad (9)$$

Thus  $R$  is the matrix that connects ‘‘free’’ incoming and outgoing bosonic fields whose dimensions we know how to compute. Now notice that when  $C^2 = \mathbb{I}$ , the above equation simplifies to  $\tilde{\phi}_{iO} = C_{ij}\tilde{\phi}_{jI}$  but not otherwise. This implies that for the case of  $C^2 = \mathbb{I}$  both the interacting fields ( $\phi_{iO/I}$ ) and the free fields ( $\tilde{\phi}_{iO/I}$ ) satisfy the same boundary condition. Also note that current conservation implies that the elements of the splitting matrix  $C$  are real and satisfy the constraint,

$$\sum_i C_{ij} = 1. \quad (10)$$

Furthermore the constraint that both the incoming and outgoing fields satisfy bosonic commutation relations independently implies<sup>9</sup>

$$\sum_j C_{ij}^2 = 1, \quad \sum_j C_{ij}C_{i+1,j} = 0, \quad (11)$$

which is essentially the same constraint that is obtained from imposing the constraint of scale invariance or requiring  $C_{ij}$  to be a fixed point.

For the three-wire system, most of the fixed points studied in Ref. 7 can be obtained as  $C$  matrices satisfying the above constraints. For instance, the disconnected normal ( $\mathbf{DN}_3$ ) fixed point where each of the wires independently has a Neumann boundary condition on the  $\phi$  field at origin corresponds to  $C = \mathbb{I}$  and the  $\mathbf{D}_P$  fixed point has  $C$  matrix of the form,

$$C = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}. \quad (12)$$

It turns out that several other  $C$  matrices obeying the constraints mentioned above fall into the two one-parameter families given in Eq. (6) and hence can be identified as conformally invariant fixed points. Also note that both the disconnected  $\mathbf{DN}_3$  and the above  $\mathbf{D}_P$  fixed points belong to the special class of  $C_{ij}$  matrix for which  $C^2 = \mathbb{I}$ .

Physically, the disconnected  $\mathbf{DN}_3$  fixed point (called  $\mathbf{N}$  in Ref. 7) corresponds to a situation where the conductance between any two wires is zero, whereas the  $\mathbf{D}_P$  fixed point corresponds to a situation where there is a perfect symmetry among the three wires, and the conductance between any two wires has the maximal value allowed by  $Z_3$  symmetry. Note that this maximum is larger than the maximal interwire conductance that would be allowed within a free-electron picture for the maximally conducting  $Z_3$  symmetric case<sup>7</sup> and this is related to the fact that for the bosonic  $Z_3$  symmetric fixed point, multiparticle scattering leads to an enhancement of conductance as was discussed in Ref. 7. In Sec. II D, by calculating the conductance, we will show that for the analogous situation in the superconducting sector, this is no longer true, and in fact, there is a reduction in the conductance as compared to the free-electron case. The difference in the processes participating in the two sectors can also be seen in Fig. 3.

The charge-conserving constraint at the junction implies that the boundary condition on the center of mass (CM) field defined as  $\phi_{\mathbf{CM}} = \sum_i \phi_i$  always has to be Neumann, i.e.,  $\sum \phi_{iI} = \sum \phi_{iO} + c$ , where  $c$  is a constant. However, in the presence of a superconducting junction strongly coupled to the wires, there will only be charge nonconserving processes at the boundary (i.e., it can either absorb or emit a Cooper-pair), and charge-conserving processes will be suppressed (at energies below the superconducting gap). Now if we impose Dirichlet boundary condition on the CM field, it turns out that it gives the correct boundary condition at the junction that converts an electron to a hole and vice versa and mimics the existence of a superconductor at the junction. This leads to new fixed points, which have not been explored in Ref. 7. This is one of the main points of our paper.

In order to establish the duality between the normal and superconducting junctions, let us consider the case of an NS junction where a single QW is connected to a superconductor (this case was considered briefly in the Appendix of Ref. 7) in the subgap regime. In the limit, when the coupling between the wire and the superconductor is strong (i.e., there is no backscattering of electrons), the system is in the perfect Andreev limit, and hence, an incoming electron current is completely reflected as an outgoing hole current, i.e.,  $j_I = -j_O$ . We call this as the Andreev ( $\mathbf{A}_1$ ) fixed point. This implies that the boundary condition on the  $\phi(x=0, t)$  field is Dirichlet [or equivalently Neumann on the dual  $\theta(x=0, t)$  field] and the total current at the junction is given by  $j = j_I - j_O = 2j_I$ . This can be easily generalized to a system of superconducting junction of  $N$  wires.

For the  $N$ -wire system, we must have the sum of the incoming electron current equal to the sum of outgoing hole current, which means that  $\sum_i j_{iI} + \sum_j j_{jO} = 0$  at the junction. In turn, this implies that  $\sum_i \rho_i = 0$ , i.e., the total electron density is zero at the junction. This is of course the correct boundary condition as the electron density is expected to vanish at the junction due to the finite superconducting gap. In terms of the splitting matrix  $C$ , the above constraint translates into the condition,

$$\sum_i C_{ij} = -1, \tag{13}$$

in contrast to the current conserving constraint [Eq. (10)].<sup>34</sup> The other constraints coming from the bosonic commutation relations that  $\phi_{iO/I}$  have to satisfy, given by Eq. (11), still remain valid. As mentioned earlier, these matrices fall into the two one-parameter families given in Eq. (6); thus enabling us to identify them as fixed points. In fact, given a  $C_{ij}$  matrix representing a fixed point in the normal sector, its dual fixed point in the superconducting sector can be obtained by transforming  $C_{ij} \rightarrow -C_{ij}$ . It can be easily checked that this prescription of finding the dual fixed points is consistent with the constraints given by Eqs. (10) and (13).

The duality between the charge conserving and the superconducting boundary conditions is now obvious and can be understood physically as follows. Current conservation implies that the net current should be zero at the junction, while superconductivity implies that the net electron density at the junction has to be zero due to the existence of the gap for single electron excitations in the superconductor. So, in the current conserving case, the boundary condition on the  $\phi_{\mathbf{CM}}$  field is Neumann [or Dirichlet on  $\theta_{\mathbf{CM}}$  field, i.e.,  $\sum_j j_i(0,t) = 0$ ], while for the superconducting case, the boundary condition is Dirichlet on  $\phi_{\mathbf{CM}}$  field [or Neumann on  $\theta_{\mathbf{CM}}$  field, i.e.,  $\sum_j \rho_j(0,t) = 0$ ]. As the  $\theta$  and the  $\phi$  fields have  $g \leftrightarrow 1/g$  duality among themselves, it automatically extends to the various fixed points in one sector and their analogs in the other sector, which are obtained by imposing further boundary conditions on the fields other than the CM field. We confirm this by explicitly calculating the scaling dimension of operators corresponding to all possible perturbations around these various fixed points.

Note however that the N-S duality exists over and above the dualities that exist within each sector. For instance, within the charge-conserving sector for the two-wire system, there exists a duality between weak backscattering (strong tunneling) and strong backscattering (weak tunneling) limits with  $g \leftrightarrow 1/g$  interchange. Similarly in the superconducting sector also there exists a duality between weak backscattering of holes or weak Andreev reflection (strong transmission of holes or strong CAR) and strong backscattering of holes or strong Andreev reflection (weak transmission of holes or weak CAR) with  $g \leftrightarrow 1/g$ . This essentially follows from the  $\mathbf{D}_{\theta_1, \theta_2}$  duality.

We will now explicitly consider the cases where there are  $N=1, 2,$  and  $3$  wires coupled to the superconductor.

### A. Single-wire junction

We start with the simplest case of the NS junction where the number of wires is  $N=1$ . In this case, there are two single element splitting matrices that satisfy the constraints of Eq. (11), and only one of them satisfies the superconducting constraint of  $C_{ij} = -1$ . In that case the wire is perfectly connected to the superconductor and an incoming electron is scattered back perfectly into a hole (see Fig. 4). This is the perfect Andreev limit described before where  $j_{iI} = -j_{iO}$ . The scaling dimension of the electron backscattering operator  $\psi_i^\dagger \psi_O$  (the subscripts  $I/O$  on the electron fields refer to incoming and

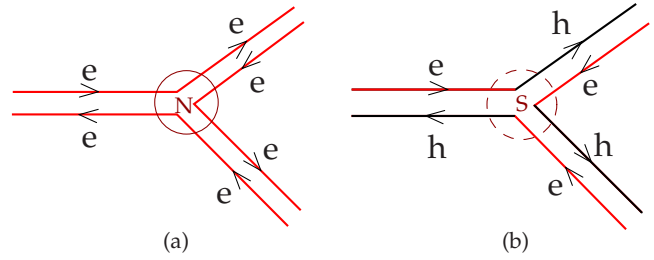


FIG. 4. (Color online) (a) Andreev and (b) normal reflections from a superconductor.

outgoing branches) around this fixed point can be easily found by bosonizing it as  $\psi_i^\dagger \psi_O \sim e^{-i\phi_I} e^{i\phi_O}$ . Upon writing it in terms of the Bogoliubov transformed fields, we can compute the scaling dimension of this operator to be  $2g$ . Note that the backscattering operator we have turned on around the charge nonconserving fixed point is charge conserving.

The other fixed point corresponds to the charge-conserving case where the splitting matrix is  $C_{ij} = +1$ . Here the incoming current is perfectly (normal) reflected ( $j_{iI} = j_{iO}$ ) and the wire is completely disconnected from the superconductor. We can now turn on a charge violating perturbation, such as the AR operator,  $\psi_I \psi_O \sim e^{i\phi_I} e^{i\phi_O}$ . The scaling dimension of this operator turns out to be  $2/g$ . This establishes the  $g \leftrightarrow 1/g$  N-S duality between these two cases.

### B. Two-wire junction

Let us now go on to the case of the NSN junction where the number of wires is  $N=2$ . In this case, the current splitting matrix is  $2 \times 2$ . Unlike the previous case (NS junction), here we find that there are two fixed points in the superconducting sector and they are represented by the following two matrices:

$$C_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

The matrix  $C_1$  corresponds to a situation where the two wires are individually tuned to the disconnected Andreev ( $\mathbf{A}_2$ ) fixed point (electrons are reflected back as holes), whereas the matrix  $C_2$  implies perfect CAR between the wires and is called the crossed Andreev  $\mathbf{CA}_2$  fixed point (electrons perfectly transmitted as holes). As can be easily checked,  $C_1$  is a particular case of  $C_{\theta_1}$  ( $\theta_1 = \pi$ ) and  $C_2$  is a particular case of  $C_{\theta_2}$  ( $\theta_2 = -\pi/2$ ) given by Eq. (6). It is easy to see that these two cases are analogous to the completely reflecting (disconnected) and completely transmitting (fully connected) cases for the normal two-wire junction.

Let us now turn on tunneling or backscattering operators as perturbations around these fixed points. Around  $C_1$ , which is fully disconnected, we switch on a CAR operator which will convert an incoming electron in one wire to an outgoing hole in another, given by  $\psi_{1I} \psi_{2O} \sim e^{i\phi_{1I}} e^{i\phi_{2O}}$ . The dimension of this operator can be computed by re-expressing the operator in terms of the Bogoliubov transformed fields.

Since the matrix  $C_1$  is just the negative of the identity matrix, it is trivial to see that the Bogoliubov transformed

fields also satisfy the same boundary conditions as the original fields. The scaling dimension can easily be computed and it turns out to be equal to  $g$ . Analogously, around the  $C_2$  fixed point, where an electron injected in the first wire gets perfectly transmitted as a hole in the second wire, we can switch on the AR operator,  $\psi_{i1}\psi_{i0} \sim e^{i\phi_{i1}}e^{i\phi_{i0}}$ . Again the bosonic fields can be re-expressed in terms of the Bogoliubov transformed  $\tilde{\phi}$  fields, and since  $C_2^2 = \mathbb{I}$ , the tilde fields also satisfy the same boundary conditions. Here, we find that the Andreev backscattering operator has the dimension  $1/g$ . Hence, within the superconducting sector, for repulsive interelectron interactions,  $g < 1$ , the  $\mathbf{CA}_2$  fixed point is a stable fixed point [shown in Fig. 2(b)], while the fully disconnected  $\mathbf{A}_2$  fixed point is unstable. This is in contrast to the normal charge-conserving junction of two wires where the ‘‘cut’’ wire [shown in Fig. 2(a)] corresponds to stable fixed point for repulsive interactions. Again this can also be understood in terms of the  $g \leftrightarrow 1/g$  N-S duality.

In the above analysis, we have restricted ourselves to either Neumann (for normal) or Dirichlet (for superconducting) boundary conditions on the CM field and then analyzed the system, which essentially reduces the system to a single boson ( $m=1$ ) problem. However, once we allow for arbitrary charge nonconservation, then for the two-wire system, both the CM field  $\frac{1}{2}(\phi_1 + \phi_2)$  and the relative field  $\frac{1}{2}(\phi_1 - \phi_2)$ , enter the picture. Hence, the system can no longer be reduced to a single boson theory as could be done when the charge conserving or the superconducting boundary condition removed the CM field completely from the scene.

Hence in general for a two-wire junction, we have a genuine  $m=N=2$  problem. As mentioned earlier, in terms of these two fields, scale invariance of the boundary condition gives us two one-parameter family of fixed points which are consistent with bosonic commutation rules imposed on the incoming and outgoing fields. The two families are connected via duality transformation ( $\phi \leftrightarrow \theta$ ) on either the  $\phi_1$  field or the  $\phi_2$  field where 1 and 2 are wire indices. So in conclusion, the important point to note is that except for  $\theta_1=0$  (cut) and  $\theta_2=\pi/2$  (healed) for the normal case or  $\theta_1=\pi$  and  $\theta_2=-\pi/2$  for the superconducting case, these fixed points belong neither to the category of charge-conserving fixed points nor to the category of superconducting fixed points. A similar isolated fixed point called Andreev-Griffiths (AG) fixed point which allowed both superconducting and charge-conserving transmissions and reflections was seen earlier in WIRG formalism by Das *et al.*<sup>10,11</sup>

### C. Three-wire junction

Finally, let us consider the case where there is a superconducting junction of three LL wires, i.e.,  $N=3$  and the current splitting matrix is  $3 \times 3$ . Here too, just as in the normal three-wire case,<sup>7</sup> we do not have a complete classification of all the fixed points of the system in general. However, a partial classification within the superconducting or the normal sector can be obtained in terms of the current splitting matrix which can be derived from the  $C_{\theta_1}$  and  $C_{\theta_2}$  matrices given in Eq. (6). For the superconducting case, it is easy to see that we will have a fixed point corresponding to the situation

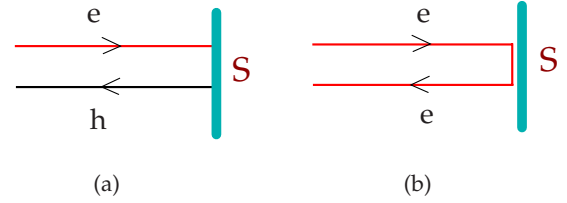


FIG. 5. (Color online) (a)  $\mathbf{DN}_3$  and (b)  $\mathbf{A}_3$  fixed points.

where each individual wire is tuned to the Andreev fixed point (see Fig. 5) at the junction. This is the disconnected Andreev ( $\mathbf{A}_3$ ) fixed point. We will also have the  $\mathbf{SD}_P$  fixed point as mentioned earlier. These are the analogs of  $\mathbf{DN}_3$  and  $\mathbf{D}_P$  fixed points for the normal sector. We will focus on these fixed points first. The current splitting matrices representing the  $\mathbf{A}_3$  and the  $\mathbf{SD}_P$  are given by

$$C_{\mathbf{A}_3}^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_{\mathbf{SD}_P}^3 = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}. \quad (15)$$

Let us now compute the stability of these two fixed points. Around  $\mathbf{A}_3$ , the CAR operator is given by  $\psi_{i1}\psi_{j0} \sim e^{i\phi_{i1}}e^{i\phi_{j0}}$ , where  $j \neq i$  and  $i, j=1, 2, 3$ . This is the same operator that we considered in the two-wire case around the  $\mathbf{A}_2$  fixed point, and the dimension of the operator of course turns out to be  $g$  because essentially it only involves tunneling between two wires that are disconnected from each other. This is dual to the scaling dimension of the normal tunneling operator around the disconnected three-wire fixed point, which is  $1/g$ . Thus, it gives a simple check of the general N-S duality.

A more nontrivial check is to consider the stability of the  $\mathbf{SD}_P$  fixed point. The general N-S duality implies that this should be dual to the usual  $\mathbf{D}_P$  fixed point of the normal junction. Let us first consider the operator corresponding to CAR, i.e.,  $\psi_{i1}\psi_{j0} \sim e^{i\phi_{i1}}e^{-i\phi_{j0}}$ . Since the matrix  $(C_{\mathbf{D}_P}^3)^2 = \mathbb{I}$ , the Bogoliubov transformed bosons also satisfy the same boundary condition as the original fields, as mentioned earlier. Hence the dimension of the operator can easily be computed to be  $1/3g$ . Now consider the scaling dimension of the normal tunneling operator,  $\psi_{i1}\psi_{j0}^\dagger \sim e^{i\phi_{i1}}e^{-i\phi_{j0}}$  around the normal  $\mathbf{D}_P$  fixed point. This has been computed in Ref. 7 to be  $g/3$ . Thus the dimensions of these operators are related by  $g \leftrightarrow 1/g$  N-S duality. Similarly, if we consider the AR operator in each wire,  $\psi_{i1}\psi_{i0} \sim e^{i\phi_{i1}}e^{i\phi_{i0}}$ , its dimension can be computed to be  $4/3g$ . This is dual to the dimension of the usual reflection operator,  $\psi_{i1}\psi_{i0}^\dagger \sim e^{i\phi_{i1}}e^{-i\phi_{i0}}$  in a normal junction which was earlier found to be  $4g/3$ .<sup>7</sup>

Finally, let us consider the tunneling of the incoming electron in wire  $i$  to the incoming electron in wire  $j$ . Here, since the tunneling happens within the incoming channels before the electron reaches the junction, the operator is given by  $\psi_{i1}\psi_{j1}^\dagger \sim e^{i\phi_{i1}}e^{-i\phi_{j1}}$ . In other words, this is a charge-conserving operator, unlike the two other (charge-violating) operators for which we calculated the scaling dimensions. Hence the third tunneling operator that we consider as a perturbation around the  $\mathbf{SD}_P$  fixed point is the same as that for the  $\mathbf{D}_P$

fixed point. However, the scaling dimension of this operator computed around the  $\mathbf{SD}_P$  and the  $\mathbf{D}_P$  fixed points turns out to be different because the boundary condition explicitly enters the computation of the scaling dimensions. They turn out to be  $1/g$  and  $g$ , respectively, which is in accord with the N-S duality.

To sum up, we find the scaling dimensions of the three classes of operators  $(\psi_{i1}\psi_{j0}, \psi_{i1}\psi_{i0}, \psi_{i1}\psi_{j1}^\dagger)$  around the  $\mathbf{SD}_P$  fixed point to be  $(1/3g, 4/3g, 1/g)$ . These are connected by N-S duality to the three classes of charge-conserving operators  $(\psi_{i1}\psi_{j0}^\dagger, \psi_{i1}\psi_{i0}^\dagger, \psi_{i1}\psi_{j1}^\dagger)$  around the  $\mathbf{D}_P$  fixed point which turn out to be  $(g/3, 4g/3, g)$ . This actually exhausts the set of all possible operators allowed by symmetry within the superconducting sector around the  $\mathbf{SD}_P$  fixed point. The important point to note is that for values of  $g$  such that  $g < 1/3$ , all these operators are irrelevant. So in the limit where the junction is tuned such that there are no charge-conserving normal tunnelings or reflections at the junction, this fixed point is stable for  $g < 1/3$ .

One can also perturbatively include the effect of charge-conserving tunneling and reflection processes at the junction and calculate their scaling dimensions. Such operators are also connected by the same N-S duality transformations between the charge conserving and the superconducting sectors. The charge-conserving tunneling operator,  $\psi_{i1}\psi_{j0} \sim e^{i\phi_{i1}}e^{-i\phi_{j0}}$ , between two wires across the superconducting junction around  $\mathbf{SD}_P$  fixed point is found to have a scaling dimension  $(2g^2+3)/3g$ . This continues to be irrelevant for  $g < 1/3$ ; hence it does not disturb the stability of the  $\mathbf{SD}_P$  fixed point. But the scaling dimension of the normal reflection around  $\mathbf{SD}_P$  fixed point turns out to be  $2g/3$ , which is relevant for  $g < 1$ . Hence, we conclude that the  $\mathbf{SD}_P$  fixed point is stable within the superconducting sector for  $g < 1/3$ , but including the normal reflection operator will make it flow to the disconnected charge-conserving fixed point. We can also view the  $\mathbf{SD}_P$  fixed point as a strong tunneling limit of the CAR processes around the  $\mathbf{A}_3$  fixed point. This is analogous to the strong coupling–weak coupling duality between the  $\mathbf{D}_P$  and the  $\mathbf{DN}_3$  fixed points for the normal three-wire system as was pointed out in Ref. 7. If we now compare the scaling dimensions of the CAR operator between the two fixed points, these are  $g$  and  $1/3g$ , respectively, for the  $\mathbf{A}_3$  and the  $\mathbf{SD}_P$  fixed points. In contrast, for the disconnected fixed point and  $\mathbf{D}_P$  fixed point of the normal wire, they are  $1/g$  and  $g/3$ , respectively. Hence the duality  $1/g \leftrightarrow g/3$  for the normal case goes over to  $g \leftrightarrow 1/3g$  in superconducting case in agreement with the N-S duality.

Next we consider the  $\mathbf{S}\chi_{\pm}$  fixed point. This fixed point is described by the following two current splitting matrices given by

$$C_{S+}^3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_{S-}^3 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \quad (16)$$

Here the subscript  $S$  stands for superconducting case and the  $\pm$  for chirality. The  $C_{S+}^3$  fixed point corresponds to a situation where there is perfect CAR of electron from wire  $1 \rightarrow 2, 2$

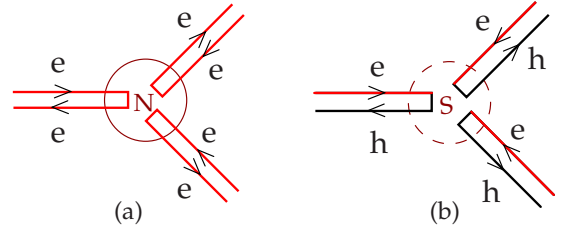


FIG. 6. (Color online) (a)  $\chi_{\pm}$  and (b)  $\mathbf{S}\chi_{\pm}$  fixed points.

$\rightarrow 3, 3 \rightarrow 1$ . On the other hand,  $C_{S-}^3$  corresponds to a situation where there is perfect CAR of electron from wire  $1 \rightarrow 3, 3 \rightarrow 2, 2 \rightarrow 1$ . The fixed point for one of these cases along with the analogous fixed point for the normal junction is shown in Fig. 6. Both these fixed points break time-reversal symmetry and depend on the direction of the effective magnetic field through the junction.

Next we calculate the scaling dimensions of all possible operators around these fixed points, which are the following: (i) AR in each wire,  $\psi_{i1}\psi_{i0} \sim e^{i\phi_{i1}}e^{i\phi_{i0}}$ ; (ii) CAR between any two wires,  $\psi_{i1}\psi_{j0} \sim e^{i\phi_{i1}}e^{i\phi_{j0}}$ ; and (iii) normal tunneling between the incoming chiral branches of any two wires,  $\psi_{i1}\psi_{j1}^\dagger \sim e^{i\phi_{i1}}e^{-i\phi_{j1}}$ . Analogous to the normal chiral case, the scaling dimensions of all these operators turn out to be the same and are given by  $4g/(3g^2+1)$  for both  $C_{S+}^3$  and  $C_{S-}^3$ . Note that all the operators listed above are marginal for  $g = 1$  and  $1/3$ . Now let us compare these with the scaling dimension of all possible operators around the chiral fixed point for the normal junction. The current splitting matrix for the normal case just requires the replacement of  $-1$  by  $1$  for both the  $C_{S+}^3$  and  $C_{S-}^3$  in Eq. (16). The scaling dimensions of all possible operators that can be switched on around either of the fixed points are given by  $4g/(3+g^2)$ . It is easy to check that the scaling dimension of the operators around the  $\mathbf{S}\chi_{\pm}$  fixed point (represented by  $C_{S+}^3$  and  $C_{S-}^3$ ) is related to that of the operators around the normal chiral fixed point by  $g \leftrightarrow 1/g$ . This is as expected from the N-S duality relation between the superconducting and the normal sectors. However, unlike the normal chiral fixed point which is stable for  $1 < g < 3$  (attractive electrons hence unphysical), the  $\mathbf{S}\chi_{\pm}$  fixed point is stable for  $1/3 < g < 1$  (repulsive electrons); this fact makes this fixed point experimentally relevant as this fixed point can be stabilized even for very weakly interacting electrons.

Now we consider the influence of charge-conserving operators corresponding to tunneling of electron  $(\psi_{i1}\psi_{j0} \sim e^{i\phi_{i1}}e^{-i\phi_{j0}})$  across the superconducting junction between any two wires and normal reflection of electrons  $(\psi_{i1}\psi_{i0} \sim e^{i\phi_{i1}}e^{-i\phi_{i0}})$  within each wire. The scaling dimensions of both these operators are given by  $[2g(1+g^2)]/(1+3g^2)$ , and hence, these operators are relevant for  $g < 1$ . This means that these fixed points are stable only within the superconducting sector for  $1/3 < g < 1$  but not in general. Hence the  $\mathbf{S}\chi_{\pm}$  fixed point can be relevant for experiments if one has weakly interacting electrons; but for strongly interacting electrons ( $g < 1/3$ ) the relevant fixed point would be the  $\mathbf{SD}_P$  fixed point as long as normal reflection at the junction is reasonably weak.

We summarize the results of this section in Table I.

TABLE I. Stability of some of the fixed points for different values of the interaction parameter  $g$  for  $N=1, 2$ , and 3 wires connected to a normal junction and a superconducting junction are tabulated.

Normal junction	Superconducting junction
(a) $N=1$ Disconnected fixed point, $\mathbf{DN}_1$ , stable for $g < 2$	(a) $N=1$ Andreev fixed point $\mathbf{A}_1$ , stable for $g < 1/2$
(b) $N=2$ Disconnected fixed point, $\mathbf{DN}_2$ , stable for $g < 1$ Fully connected fixed point, stable for $g > 1$	(b) $N=2$ Andreev fixed point $\mathbf{A}_2$ , stable for $g > 1$ Crossed Andreev fixed point, $\mathbf{CA}_2$ , stable for $g < 1$
(c) $N=3$ Disconnected fixed point, $\mathbf{DN}_3$ , stable for $g < 1$ Chiral fixed point, $\chi_{\pm}$ , stable for $1 < g < 3$ $\mathbf{D}_p$ fixed point, stable for $g > 3$	(c) $N=3$ Andreev fixed point, $\mathbf{A}_3$ , stable for $g > 1$ Supercond. chiral fixed point, $\mathbf{S}\chi_{\pm}$ , stable for $1/3 < g < 1$ $\mathbf{SD}_p$ fixed point, stable for $g < 1/3$

#### D. Conductance matrix and N-S duality

As shown explicitly in the Appendix of Ref. 26, the conductance tensor relating the current and voltage in different wires can be computed from the Kubo formula and can be related to correlation functions of the incoming and outgoing currents. These correlation functions can be computed in terms of the  $R_{ij}(g)$  matrix relating the incoming and outgoing free fields. Note from Eq. (9), that this matrix is in general a function of  $g$ .

So far, we have discussed the dualities existing between the fixed points in the normal and superconducting sectors in terms of scaling dimensions of various perturbations around the fixed points. Now we can quantify the duality in terms of the fixed-point conductances of the various fixed points in the normal sector and the corresponding dual fixed points in the superconducting sector.

Let  $R^N$  represent the  $R$  matrix [which connects the free incoming and outgoing fields and which has been defined in Eq. (9)] for a fixed point in the normal sector. Then the  $R$  matrix corresponding to the dual fixed point in the superconducting sector is given by

$$R^S(g) = -R^N(1/g). \quad (17)$$

This can explicitly be checked from Eq. (9) when we make the duality transformation,  $C_{ij} \rightarrow -C_{ij}$ . In terms of the  $R^N$  and  $R^S$  matrices, the conductance matrix is given by<sup>26</sup>

$$G^N = \frac{ge^2}{h}(\mathbb{I} - R^N), \quad G^S = \frac{ge^2}{h}(\mathbb{I} - R^S), \quad (18)$$

where  $N$  and  $S$  stand for normal and superconducting cases, respectively, and the identity factor just represents the incoming current along each wire. For the two-wire (NSN) junction the stable fixed point within the superconducting sector for  $g < 1$  was found to be the  $\mathbf{A}_2$  fixed point. The  $R$  matrix is identical to the  $C$  matrix for this case as  $C^2 = \mathbb{I}$ , and hence,

$$G_{\mathbf{A}_2}^S = \frac{ge^2}{h} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (19)$$

This implies that if the two wires (labeled 1 and 2) are biased with respect to the superconductor at voltages  $V_1$  and  $V_2$ ,

then for an injected electron current,  $(e^2/h)V_1$ , the junction will suck in an electron current equal to  $(e^2/h)V_2$  from wire 2. Next let us consider the  $\mathbf{SD}_p$  fixed point for the three-wire case. Here also as  $C^2 = \mathbb{I}$ , the  $R$  matrix is identical to the  $C$  matrix, and hence the conductance can be directly written down using Eq. (18),

$$G_{\mathbf{SD}_p}^S = \frac{ge^2}{h} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (20)$$

This is to be contrasted with the conductance matrix for the  $Z_3$  symmetric AG fixed point<sup>10</sup> for the three-wire system within the free-electron manifold obtained using WIRG method given by

$$G_{\text{free-electron}}^S = \frac{e^2}{h}(\mathbb{I} + \tilde{S}) = \frac{e^2}{h} \begin{pmatrix} 10/9 & 4/9 & 4/9 \\ 4/9 & 10/9 & 4/9 \\ 4/9 & 4/9 & 10/9 \end{pmatrix}, \quad (21)$$

where the elements of  $\tilde{S}$  are obtained from the  $S$  matrix for the free-electron problem by taking squares of the absolute values of individual elements themselves. This implies that  $\tilde{S}$  is just the current splitting matrix for the free-electron case.

In Ref. 7, it was pointed out that for the normal junction, the diagonal conductance for the  $\mathbf{D}_p$  fixed point with Fermi liquid leads,  $(G_{\mathbf{D}_p}^S)_{ii} = 4e^2/3h$ , was larger than its free-electron counterpart  $8e^2/9h$  for the Griffiths fixed point.<sup>5</sup> In contrast, we find that this is no longer true for the superconducting case. The free-electron conductance for the AG fixed point is  $10e^2/9h$ , which is actually larger than the diagonal conductance for the  $\mathbf{SD}_p$  fixed point with Fermi liquid leads given by  $(G_{\mathbf{SD}_p}^S)_{ii} = e^2/h$ . So in conclusion, electron-electron interactions lead to either enhancement or suppression of conductance with respect to its free-electron counterpart depending on whether the junction is normal or superconducting.

Finally, we consider the conductance matrix for the  $\mathbf{S}\chi_{\pm}$  fixed point. For this case also, one can compute the conductance by evaluating the  $R$  matrices corresponding to the  $C_{S+}^3$  and  $C_{S-}^3$  [Eq. (16)] which are



$$\mathbb{R}_{C_{S+}^3} = \begin{pmatrix} \frac{-1+g^2}{1+3g^2} & -\frac{2g(1+g)}{1+3g^2} & -\frac{2(-1+g)g}{1+3g^2} \\ -\frac{2(-1+g)g}{1+3g^2} & \frac{-1+g^2}{1+3g^2} & -\frac{2g(1+g)}{1+3g^2} \\ -\frac{2g(1+g)}{1+3g^2} & -\frac{2(-1+g)g}{1+3g^2} & \frac{-1+g^2}{1+3g^2} \end{pmatrix}, \quad (22)$$

$$\mathbb{R}_{C_{S-}^3} = \begin{pmatrix} \frac{-1+g^2}{1+3g^2} & -\frac{2(-1+g)g}{1+3g^2} & -\frac{2g(1+g)}{1+3g^2} \\ -\frac{2g(1+g)}{1+3g^2} & \frac{-1+g^2}{1+3g^2} & -\frac{2(-1+g)g}{1+3g^2} \\ -\frac{2(-1+g)g}{1+3g^2} & -\frac{2g(1+g)}{1+3g^2} & \frac{-1+g^2}{1+3g^2} \end{pmatrix}. \quad (23)$$

Now the conductance matrix immediately follows from Eq. (18). Unlike the case of  $\mathbf{SD}_P$  fixed point, the  $\mathbb{R}$  matrix in this case depends on the LL parameter  $g$ . Given this, one can now explicitly check the duality as defined in Eq. (17) between the chiral fixed points for the normal junction and the  $\mathbf{S}\chi_{\pm}$  fixed points in the superconducting sector by comparing the conductance matrix for  $\mathbf{S}\chi_{\pm}$  fixed points with the conductance matrix obtained in Ref. 7 for the normal chiral fixed points.

Hence with all the explicit checks via the scaling dimensions of operators and conductance calculations, we have established N-S duality rigorously for junctions comprising of one, two, and three wires. In general, the N-S duality holds for junctions of any number of wires including those with  $N > 3$ .

### III. DISCUSSION

In the context of three wires, both within the superconducting sector and in the normal sector, there are only two independent bosonic fields connected to the junction since the CM field is constrained either by charge conservation or by the superconducting boundary condition. As in the superconducting two-wire junction, this implies that scale invariance of the boundary condition leads to two one-parameter families of fixed points, which are related by a  $\mathbf{D}_{\theta_1\theta_2}$  duality transformation on one of the two independent fields. But for the three-wire system the  $\mathbf{D}_{\theta_1\theta_2}$  duality transformation is not a one to one map between the two one-parameter families of fixed points. For example, for the superconducting three-wire junction, when the  $\mathbf{D}_{\theta_1\theta_2}$  duality transformation is made on the  $\frac{1}{2}(\phi_1 - \phi_2)$  field, the  $\mathbf{A}_3$  fixed point (a representative of the  $C_{\theta_1}$  family) goes to the  $\mathbf{SD}_A$  fixed point (a  $Z_3$  asymmetric fixed point where two of the three wires are stuck at the  $CA_2$

fixed point and the third wire is tuned to the  $\mathbf{A}_1$  fixed point with the junction) fixed point which is in the  $C_{\theta_2}$  family. But when the duality transformation is made on the  $\frac{1}{6}(\phi_1 + \phi_2 - 2\phi_3)$  field, the disconnected fixed point goes to the fixed point given by the matrix,

$$C_{D_P}^3 = \begin{pmatrix} -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & -2/3 \end{pmatrix}, \quad (24)$$

in the  $C_{\theta_2}$  family.

One of the most intriguing observations in the context of junctions of LL in general is that in the  $g \rightarrow 1$  limit, all the fixed points in the bosonic language which have entries zero and unity in the current splitting matrix reduce to the fermionic fixed points which have entries zero and unity in the current splitting matrix (obtained from the free-electron  $S$  matrix) which were obtained using WIRG formalism. The scaling dimensions of all possible perturbations around these fixed points also match in this limit. But this is not generally true when the entries are not 0 and 1. For instance, the  $\mathbf{D}_P$  fixed point in bosonic language is not the same as the fermionic Griffiths fixed point in the WIRG formalism, as neither the conductance at the fixed point nor the scaling dimensions of the operators around the fixed point match in the  $g \rightarrow 1$  limit even though they share the same  $Z_3$  symmetry, and the current splitting matrix for the  $\mathbf{D}_P$  fixed point is identical to the  $S$  matrix for the fermionic Griffiths fixed point.

Also from the N-S duality, we should expect the analogs of the  $\mathbf{D}_N$  and the  $\mathbf{M}$  fixed points of Refs. 6 and 7 to exist in the superconducting three-wire system, but a more detailed analysis of these fixed points is beyond the scope of the present work.

Last but not the least it is very interesting to note that all the fixed points that we have considered in this paper are noiseless (because there is no probabilistic partitioning of the current) even though many of them do not correspond to the perfectly transmitting or perfectly reflecting situations. This is unlike the fermionic fixed point obtained using WIRG formalism where the Griffiths fixed point and the Andreev-Griffiths fixed points had probabilistic partitioning of the current and were noisy.

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\*sourin@cts.iisc.ernet.in

†sumathi@hri.res.in

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